A short course on Harnack inequality for elliptic PDEs

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1 introduction

Let Ω be a open set in $\mathbb{R}^n$. Denote L be a elliptic operator in divergence form. That is

$$Lu = \sum_{ij} \partial_i(a_{ij}\partial_j u)$$

where $a_{ij}(x)$ are measurable function. Furthermore, we assume a_{ij} satisfies

$$\lambda^{-1}|z|^2 \leq a_{ij}(x)z_i z_j \leq \lambda|z|^2 \quad, \forall\, x \in \Omega, \forall z \in \mathbb{R}^n.$$

We say u is a weak solution of $Lu = 0$ if $\int_\Omega a_{ij}\partial_i u \partial_j \phi = 0$ for any $\phi \in C_0^\infty(\Omega)$. So the natural class of the solutions is $W^{1,2}_{loc}(\Omega)$.

Theorem 1.1.
(De-Giorgi) If u is a solution of $Lu = 0$, then $u \in C^\alpha$.
(J-Nash) Same result for the parabolic case.
(J-Moser) Harnack inequality holds for solution of $Lu = 0$. In particular, u is Holder continuous.

Analogously, we can consider L to be in non-divergence form, $L = a_{ij}\partial_i\partial_j$. Then the natural case of solution will be $W^{2,2}(\Omega)$.

Theorem 1.2. *(Krylov-Safonov) u is C^α in the non-divergence case (including parabolic case). Harnack inequality also holds.*

In this note, we will focus on the elliptic PDE case. All sup and inf are understood to be essential supremum and essential infimum if not specified in the content.

2 Hölder Inequality for divergence form

In this section, we assume $L = \partial_i(a_{ij}\partial_j)$ where a_{ij} is measurable, symmetric and uniformly elliptic by a constant $\lambda > 0$.

Theorem 2.1. *(Mean value inequality) Let $u \in W^{1,2}(B_R)$, $Lu \geq 0$. Then there exists $C = C(n, \lambda) > 0$, such that*

$$\sup_{B_{R/2}} u_+ \leq \frac{C}{R^{n/2}}||u_+||_{L^2(B_R)}$$

Proof. **Claim:** For $f \in W^{1,2}(\mathbb{R}^n), f \geq 0$. Denote $F = \{f > 0\}$. Then

$$||f||_{L^2} \leq C_n|F|^{1/n}||\nabla f||_{L^2}.$$

Let $v = f^2$, by Sobolev inequality,

$$\left(\int_{\mathbb{R}^n} v^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C_n \int_{\mathbb{R}^n} |\nabla v|$$

So,

$$||f||_2^2 = \int_F v \leq \left(\int_F v^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} |F|^{1/n} \leq C_n |F|^{1/n} \int_{\mathbb{R}^n} |\nabla v|$$
$$= C_n |F|^{1/n} \int_{\mathbb{R}^n} f|\nabla f| \leq C_n |F|^{1/n} ||f||_2 ||\nabla f||_2,$$

which implies our claim.
For $0 < r < \rho < R$, $0 < \alpha < \beta$. Denote

$$a = \int_{B_\rho} (u-\alpha)_+^2 \quad \geq \int_{B_\rho} (u-\beta)_+^2 = b.$$

Let $v = (u-\beta)_+$. Choose a cut-off function η such that $\eta = 1$ on $B(r)$ and $\eta = 0$ outside $B(\rho)$. In particular, we may assume $|\nabla \eta| \leq \frac{2}{\rho - r}$. By approximation using smooth function, we have

$$\int a_{ij} \partial_j u \partial_i v \cdot \eta^2 \leq -2 \int a_{ij} \partial_j u \partial_i \eta \cdot v\eta$$

by putting $\phi = v\eta^2$. We also have $\partial_i v \partial_j u = \partial_i v \partial_j v$ and $v\partial_j u = v\partial_j v$. Substitute it back to above inequality to yield

$$\int a_{ij} \partial_j v \partial_i v \cdot \eta^2 \leq -2 \int a_{ij} \partial_j v \partial_i \eta \cdot v\eta \leq 2\lambda \int v\eta |\nabla v||\nabla \eta|$$
$$\leq 2\lambda \left(\int |\nabla v|^2 \eta^2\right)^{1/2} \left(\int |\nabla \eta|^2 v^2\right)^{1/2}$$

By uniform ellipticity, we have

$$\int |\nabla v|^2 \eta^2 \leq 4\lambda^4 \int v^2 |\nabla \eta|^2.$$

By our choice of η,

$$\int |\nabla (v\eta)|^2 \leq 2 \int \eta^2 |\nabla v|^2 + v^2 |\nabla \eta|^2 \leq 2(4\lambda^4 + 1) \int v^2 |\nabla \eta|^2 \leq \frac{C}{(\rho - r)^2} \int_{B_\rho} v^2.$$

For $F' = \{u > \beta\} \cap B_\rho$,

$$a = \int_{B_\rho} (u-\alpha)_+^2 \geq \int_{F'} (u-\alpha)_+^2 \geq (\beta - \alpha)^2 |F'| \implies |F'| \leq a(\beta - \alpha)^{-2}.$$

By our claim, for $F = \{v\eta > 0\}$

$$||v||_{L^2(B_r)} \leq ||v\eta||_{L^2} \leq C|F|^{1/n} ||\nabla (v\eta)||_{L^2} \leq |F'|^{1/n} \frac{C}{\rho - r} ||v||_{L^2(B_\rho)}$$
$$\leq \frac{C\sqrt{a}}{\rho - r} \left[\frac{a}{(\beta - \alpha)^2}\right]^{1/n}.$$

That is to say

$$b \leq \frac{a^{1+2/n}}{(\beta-\alpha)^{4/n}} \frac{C}{(\rho-r)^2}.$$

Choose $R_k = R(2^{-1}+2^{-k})$, $\alpha_k = \alpha(2-2^{-k})$, where α is to be determined later. Let $a_k = \int_{B_{R_k}} (u-\alpha_k)_+^2$ and denote $q = 1+2/n$. By putting $b = a_k, a = a_{k-1}, r = R_k, \rho = R_{k-1}$, $\beta = \alpha_k$,$\alpha = \alpha_{k-1}$ into the above inequality, one can show that

$$a_k \leq \frac{Ca_{k-1}^q}{\alpha^{4/n}R^2} = \frac{Ca_{k-1}^q}{M}.$$

By induction on k, we can obtain the followings.

$$a_k \leq a_0^{q^k} \cdot \frac{C^{k+q(k-1)+q^2(k-2)+\ldots+q^{k-1}}}{M^{1+q+q^2+\ldots q^{k-1}}} = a_0^{q^k} \left[\frac{C^{\frac{q^{k+1}-(k+1)q+k}{(q-1)^2}}}{M^{\frac{q^k-1}{q-1}}} \right] \leq \left[\frac{a_0 C^{\frac{2q}{(q-1)^2}}}{M^{\frac{1}{(q-1)}}} \right]^{q^k}.$$

Noted that a_0 is bounded from above by $\int_{B_R} u_+^2$. Thus, we may choose

$$\alpha = \left[\frac{2C^{\frac{2q}{(q-1)^2}} \int_{B_R} u_+^2}{R^n} \right]^{1/2}$$

which implies that $a_k \to 0$ as $k \to \infty$. Therefore, $\int_{B_{R/2}} (u-\alpha)_+^2 = 0$ which implies the conclusion.

□

Before proceed to the weak Harnack inequality, we introduce a version of Poincare ineqaulity.

Lemma 2.2. *(Poincare inequality) For $r \leq 3R$, $H = \{v \leq 0\} \cap B_r$. For all $v \in W^{1,1}(B_r)$, we have*

$$\int_{B_r} v_+^2 \leq \frac{Cr^2|B_r|}{|H|} \int_{B_r} |\nabla v_+|^2.$$

Proof. Let $u = v_+$, by standard Poincare inequality, we have

$$\int_{B_r} |\nabla u|^2 \geq \frac{C_n}{r^2} \int_{B_r} |u-\bar{u}|^2$$

where $\bar{u} = \frac{1}{V(r)} \int_{B_r} u$. Thus,

$$\int_{B_r} |\nabla u|^2 \geq \frac{C_n}{r^2} \int_H |u-\bar{u}|^2 = \frac{C_n|H|}{r^2}|\bar{u}|^2 = \frac{C_n}{r^2}\frac{|H|}{B_r} \int_{B_r} |\bar{u}|^2$$

and

$$\int_{B_r} |\nabla u|^2 \geq \frac{C_n}{r^2} \int_{B_r} |u-\bar{u}|^2 \geq \frac{C_n}{r^2}\frac{|H|}{B_r} \int_{B_r} |u-\bar{u}|^2.$$

Summing them up to yield

$$\int_{B_r} |\nabla u|^2 \geq \frac{C_n}{2r^2}\frac{|H|}{B_r} \int_{B_r} (|\bar{u}|^2 + |u-\bar{u}|^2) \geq \frac{C_n}{2r^2}\frac{|H|}{B_r} \int_{B_r} |u|^2.$$

□

Theorem 2.3. *(weak Harnack inequality) Let u be a solution of $Lu = 0$ in B_{3R}, $u \geq 0$. For any $a > 0$ $E = \{u \geq a\} \cap B_R$, then $\forall \epsilon > 0$, there exists $\delta = \delta(\epsilon, \lambda, n) > 0$ such that whenever $\frac{|E|}{|B_R|} \geq \epsilon$, then $\sup_{B_R} u \geq \delta a$.*

Proof. By adding a constant, we may assume $ess \inf u > 0$. By rescaling, we may also assume $a = 1$. Let $v = -\log u$. v is bounded from above and locally bounded from below according to the mean value inequality.

Claim: $Lv \geq 0$**.**

For any $\eta \in C_0^\infty(B_{3R})$, $\eta \geq 0$.

$$-\int a_{ij}\partial_i v \partial_j \eta = \int a_{ij}\frac{\partial_i u}{u}\partial_j \eta = \int a_{ij}\partial_i u \partial_j(\eta/u) + \int a_{ij}\partial_i u \cdot \eta \partial_j u \cdot u^{-1} \geq 0.$$

Using the special version of Poincare inequality, we can estimate the essential sup norm of v_+ by its $W^{1,2}$ norm.

$$V(2R) \cdot \sup_{B_R} v_+^2 \leq C\int_{B_{2R}} v_+^2 \leq CR^2\frac{|B_{2R}|}{|E|}\int_{B_{2R}} |\nabla v_+|^2 \leq \frac{CR^2}{\epsilon}\int_{B_{2R}} |\nabla v_+|^2.$$

On the other hand, choose a cut-off function η such that $\eta = 1$ on B_{2R}, vanishes outside B_{3R} and $|\nabla \eta| = O(1/R)$. By uniform ellipticity,

$$\lambda^{-1}\int_{B(3R)} \eta^2|\nabla v|^2 \leq \int a_{ij}\partial_j v \partial_i v \cdot \eta^2 = -\int a_{ij}\partial_j v \partial_i(\eta^2) \leq 2\lambda(\int |\nabla \eta|^2)^{1/2}(\int \eta^2|\nabla v|^2)^{1/2}.$$

Implying

$$\int_{B_{2R}} |\nabla v|^2 \leq CR^{n-2}.$$

Combining everything, $\sup_{B_R} v_+^2 \leq C/\epsilon$. That is to say $\inf_{B_R} u \geq \delta = \exp(-C/\epsilon^{1/2})$. $\square$

To show the Holder continuity, we first show a oscillation inequality.

Theorem 2.4. *Let $Lu = 0$ on B_{3R}, then $osc_{B_R} u \leq \gamma \cdot osc_{B(3R)} u$, where $osc_B\, f = \sup_B f - \inf_B f$ and $\gamma = \gamma(n, \lambda) < 1$.*

Proof. By scaling and translation, we may assume $\inf_{B(3R)} u = 0$ and $\sup_{B(3R)} u = 2$. Consider $\{u \geq 1\} \cap B(R)$ and $\{u \leq 1\} \cap B(R)$. At least one of them is of measure greater than $\frac{1}{2}|B(R)|$.

Suppose $\{u \geq 1\} \cap B(R)$ has measure greater than $\frac{1}{2}|B(R)|$. Applying the weak Harnack with $a = 1$, $\epsilon = 1/2$. We have $\inf_{B(R)} u \geq \delta = \delta(n, \lambda)$. Thus,

$$osc_{B(R)} u = \sup_{B(R)} u - \inf_{B(R)} u \leq 2 - \delta = \left(\frac{2-\delta}{2}\right) \cdot osc_{B(3R)} u = \gamma \cdot osc_{B(3R)} u.$$

If $\{u \leq 1\} \cap B(R)$ has measure greater than $\frac{1}{2}|B(R)|$, we then consider $v = 2 - u$ and repeat the argument above to obtain

$$osc_{B(R)} u = osc_{B(R)} v \leq \gamma \cdot osc_{B(3R)} v = \gamma \cdot osc_{B(3R)} u.$$

$\square$

We now are capable of showing the result by De-Gorgi.

Theorem 2.5. *Let $Lu = 0$ on $Omega \subset \mathbb{R}^n$ where $u \in W^{1,2}_{loc}(\Omega)$. Then $u \in C^\alpha(\Omega)$ for some $\alpha = \alpha(n, \lambda) > 0$. Moreover, for any compact set $K \subset\subset \Omega$, we have*

$$||u||_{C^\alpha(K)} \leq C||u||_{L^2(\Omega)}$$

where $C = C(n, K, \Omega, \lambda)$.

Proof. Let $\rho = dist(K, \partial\Omega) > 0$ and $\rho_k = 3^{-k}\rho$. For all $z \in K$, by Theorem 2.4,

$$osc_{B(z,\rho_k)} u \leq \gamma \cdot osc_{B(z,\rho_{k-1})} u \leq \gamma^{k-1} osc_{B(z,\rho_1)} u \leq 2\gamma^{k-1}||u||_{L^\infty(B(z,\rho_1))}.$$

By mean value inequality, we further conclude that

$$osc_{B(z,\rho_k)} u \leq C_{n,\rho}\gamma^k||u||_{L^2(\Omega)}.$$

Lemma 2.6. *For almost all $x, y \in K$ with $|x-y| \leq \rho/2$, $|u(x)-u(y)| \leq C|x-y|^\alpha||u||_{L^2(\Omega)}$, where $\alpha = -\log_3 \gamma > 0$.*

Proof. For $x, y \in K$, there exists $k \in \mathbb{N}$ such that $\frac{\rho_{k+1}}{2} \leq |x-y| \leq \frac{\rho_k}{2}$. Cover K by finite number of Balls $B(z_i, \rho_k/2)$, $z_i \in K$. $x \in B(z_j, \rho_k/2)$ for some j and $y \in B(z_j, \rho_k)$. For almost all such $x, y \in B(z_j, \rho_k)$,

$$|u(x) - u(y)| \leq C\gamma^k||u||_{L^2(\Omega)}.$$

But $|x-y| \geq 3^{-k-1}\rho$ implies $\gamma^k \leq \gamma^{-1}\left(\frac{2|x-y|}{\rho}\right)^\alpha$. Substitute it back to obtain the desired result. □

Lemma 2.7. *There exists $\tilde{u} \in C^\alpha(K)$ such that $|\tilde{u}(x) - \tilde{u}(y)| \leq C|x-y|^\alpha||u||_{L^2(\Omega)}$ and $\tilde{u} = u$ almost everywhere.*

Proof. For $x \in \Omega$, $r > 0$, define $u_r(x) = \frac{1}{|B_r|}\int_{B_r(x)} u$. For $x \in K$, we now show that $\{u_r(x)\}_{r>0}$ is cauchy. Write u_r and u_R as follows

$$u_r(x) = \frac{1}{|B_r|}\int_{B_r(x)} u(\xi)\, d\xi = \frac{1}{|B_r||B_R|}\int_{B_r(x)}\int_{B_R(x)} u(\xi)\, d\xi d\eta$$

and

$$u_R(x) = \frac{1}{|B_R|}\int_{B_R(x)} u(\eta)\, d\eta = \frac{1}{|B_r||B_R|}\int_{B_r(x)}\int_{B_R(x)} u(\eta)\, d\xi d\eta.$$

Thus, for $R > r$

$$\begin{aligned} |u_R(x) - u_r(x)| &\leq \frac{1}{|B_r||B_R|}\int_{B_r(x)}\int_{B_R(x)} |u(\xi) - u(\eta)|\, d\xi d\eta \\ &\leq \frac{C||u||_{L^2}}{|B_r||B_R|}\int_{B_r(x)}\int_{B_R(x)} |\eta - \xi|^\alpha\, d\xi d\eta \\ &\leq CR^\alpha||u||_{L^2} \to 0 \text{ as } R \to 0. \end{aligned}$$

So $\tilde{u} = \lim_r u_r(x)$ exists for $x \in K$. By lebesgue theorem, $\tilde{u} = u$ almost everywhere. It remains to show the Holder continuity. Similar to above, we have for all $x, y \in K$ and $|x - y| \leq \rho/4$

$$
\begin{aligned}
|u_r(x) - u_r(y)| &\leq \frac{1}{|B_r|^2} \int_{B_r(x)} \int_{B_r(y)} |u(\xi) - u(\eta)| \, d\xi d\eta \\
&\leq \frac{C||u||_{L^2}}{|B_r|^2} \int_{B_r(x)} \int_{B_r(y)} |\eta - \xi|^\alpha \, d\xi d\eta \\
&\leq C||u||_{L^2}(|x - y| + 2r)^\alpha.
\end{aligned}
$$

Taking $r \to 0$ to conclude this. □

It remains to show that the $||u||_{C^\alpha(K)}$ is controlled by the L^2-norm on Ω. For $x \in K$, by mean value inequality

$$|\tilde{u}(x)| \leq \sup_{B(x,\rho/2)} |\tilde{u}| \leq C||u||_{L^2(B(x,\rho))} \leq C||u||_{L^2}.$$

The Holder norm follows from the above lemma and the fact that $\tilde{u}$ is bounded in K. □

3 H$\ddot{o}$lder Inequality for non-divergence form

In this section, we denote $L = \sum_{i,j} a_{ij} \partial_i \partial_j$ where a_{ij} are measurable, symmetric and uniformly elliptic with ellipticity constant $\lambda > 0$.

Theorem 3.1. *(Krylov-Safonov) If $Lu = 0$ in Ω where $u \in W^{2,p}_{loc}$, then $u \in C^\alpha$ for some $\alpha = \alpha(n, \lambda) > 0$. Moreover, for any compact set K in Ω, we have the following estimate*

$$||u||_{C^\alpha(K)} \leq C||u||_{W^{2,p}(\Omega)}$$

where $C = C(n, \lambda, K, \Omega) > 0$.

We first show a smooth version of the estimate.

Theorem 3.2. *Suppose in addition $a_{ij} \in C^\infty(\Omega)$, and u is classical solution. Then we can find a $\alpha = \alpha(n, \lambda) > 0$ such that for any compact set $K \subset \Omega$, $\exists C = C(n, \lambda, K, \Omega) > 0$ such that*

$$||u||_{C^\alpha(K)} \leq C||u||_{C(\Omega)}.$$

Remarks: If $p > n$, it can be easily seen that Theorem (3.2) will imply Theorem (3.1) by approximation arguement and Sobolev ineqaulity. For $p \leq n$ case, it is claimed to be still true by GRIGOR'YAN(??).

Before we proceed to the proof of Theorem 3.2, we show the weak Harnack for the case of non-divergence operator first.

Theorem 3.3. *(weak Harnack inequality) Let u be a solution of $Lu = 0$ in B_{4R}, $u \geq 0$. For any $a > 0$ $E = \{u \geq a\} \cap B_R$, then $\forall \theta > 0$, there exists $\delta = \delta(\theta, \lambda, n) > 0$ such that whenever $\frac{|E|}{|B_R|} \geq \theta$, then $\sup_{B_R} u \geq \delta a$.*

Proof. By scaling, we may assume $a = 1$.

Lemma 3.4. *If E contain a ball of radius $\rho > 0$, then $\inf_{B_R} u \geq c\left(\frac{\rho}{R}\right)^s$ where c, s are constants depending on n and λ only.*

Proof. By translation, we assume the ball contained in E is centred at origin and u is a solution of $Lu = 0$ on $B(z, 4R)$. Let $G = \{u < 1\}$ in $B(z, 4R)$. We would like to construct a barrier function $w(x)$ on G in order to estimate u.
Our goal is to construct $w(x)$ such that $Lw \geq 0$ on G, $w \leq 0$ on $\partial B(z, 4R)$ and $w \leq 1$ on G. If such function $w(x)$ exists, $Lw \geq Lu = 0$ and

$$w(x) \leq 1 = u(x) \text{ on } \partial G.$$

By maximum principle, $w \leq u$ on G.

It remains to constuct w. Consider the function $|x|^{-s}$ where s is to be determined. Direct computation yield

$$L\left(\frac{1}{|x|^s}\right) = s|x|^{-s-2}\left((s+2)\sum_{i,j} a_{ij}\frac{x_i x_j}{|x|^2} - \sum_i a_{ii}\right) \geq s|x|^{-s-2}(\frac{s}{\lambda} - \lambda n).$$

So $L(|x|^{-s}) \geq 0$ if we choose $s = 2\lambda^2 n$. Choose $w(x) = \rho^s\left[\frac{1}{|x|^s} - \frac{1}{(3R)^s}\right]$. Clearly, $Lw \geq 0$. As $\rho \leq |x|$ on G, $w(x) \leq 1$ on G. On $\partial B(z, 4R)$, $|x| \geq |z| - |x - z| = 4R - R = 3R$.

$$w(x) = \rho^s(\frac{1}{|x|^s} - \frac{1}{(3R)^s}) \leq 0.$$

On $G \cap B(z, R)$, $|x| \leq |x - z| + |z| \leq 2R$.

$$w(x) \geq \rho^s\left[\frac{1}{(2R)^s} - \frac{1}{(3R)^s}\right] = C\left[\frac{\rho}{R}\right]^s.$$

Therefore, $\inf_{B(z,R)} u = \inf_{B(z,R)\cap G} u \geq \inf_{B(z,R)\cap G} w \geq C\left[\frac{\rho}{R}\right]^s$. □

Lemma 3.5. *If $\dfrac{|G|}{|B(4R)|} < \epsilon = \epsilon(n, \lambda)$ where $G = \{u > 1\}$, then $\inf_{B(R)} u \geq \frac{1}{2}$.*

Proof. Choose G' such that $\dfrac{|G'|}{|B(4R)|} < \epsilon$. Find $f \in C^\infty(\overline{B(4R)})$ such that $f = 1$ on G and *supp*$(f) \subset \overline{G'}$. Solving the Dirichlet problem $Lv = -f$ on $B(4R)$ and $v = 0$ on $\partial B(4R)$. v is classical, and $v \geq 0$ by maximal principle. Also by Aleksandrov-Pucci estimate,

$$\sup_{B(4R)} v \leq C_{n,\lambda} R||f||_{L^n(B(4R))}.$$

f is supported in G' and $f \le 1$,

$$\sup_{B(4R)} v \le CR||f||_{L^n(B(4R)} \le CR|G'|^{1/n} \le C_0 R^2 \epsilon^{1/n}.$$

Define $w(x) = c_1 - c_2|x|^2 - c_3 v(x)$, where c_i is to be found.

$$Lw = -2c_2 \sum_i a_{ii} + c_3 f \ge -2c_2 n\lambda + c_3 \text{ on } G.$$

On $\partial B(4R)$, $w \le c_1 - (4R)^2 c_2$. Now we choose $c_1 = 1$, $c_2 = (4R)^{-2}$ and $c_3 = n\lambda(8R^2)^{-1}$. Then $Lw \ge 0$, $w|_{\partial B(4R)} \le 0$ and $w|_G \le 1$. Maximum principle implies $u \ge w$ on G. In particular,

$$\inf_{B_R} u = \inf_{B_R \cap G} u \ge \inf_{B_R \cap G} w \ge c_1 - c_2 R^2 - \sup_{B_R} v \cdot c_3 = 1 - \frac{1}{16} - \frac{n\lambda}{8} C_0 \epsilon^{1/n}.$$

If ϵ is small enough, then $u \ge \frac{1}{2}$. □

Lemma 3.6. *If $|G \cap B_R| < \epsilon |B_R|$, then $\inf_{B_R} u \ge \gamma(n, \lambda)$. $\epsilon = \epsilon(n, \lambda)$ specified in the lemma 3.5.*

Proof. By lemma (3.5), we have $\inf_{B(R/4} u \ge \frac{1}{2}$. Hence $\{2u \ge 1\}$ contains a ball of radius $R/4$. By lemma 3.4,

$$\inf_{B_R} 2u \ge c \left(\frac{R/4}{R} \right)^s = \frac{c}{4^s} \ge \gamma.$$

□

Define $E_k = \{u \ge \gamma^k\} \cap B_R$. E_k is increasing sequence.
Main claim: For all $k \in \mathbb{N}$, either $|E_{k+1}| \ge (1+\beta)|E_k|$ for $\beta = \beta(n, \lambda) > 0$ or $E_{k+l} = B_R$ for some $l = l(n, \lambda, \theta)$.
If the main claim is true, there exists minimal $k = N$ such that $|E_N| \ge (1+\beta)|E_{N-1}| \ge (1+\beta)^N |E_0| \ge (1+\beta)^N \theta |B_R|$. And $E_{N+l} = B_R$. That is to say on B_R,

$$u \ge \gamma^{N+l} = \gamma^{l + \frac{-\log\theta}{\log(1+\beta)}} = \delta(n, \lambda, \theta).$$

It remains to show the main claim.

By considering $v = u\gamma^{-k}$, it suffices to show the situation $k = 0$. When $k = 0$, $E_0 = E$, there exists $\rho \in (0, R)$ such that $|E \cap B_{R-\rho}| = |E|/2$. Denote $F = \{u \ge 1\} \cap B_{R-\rho}$, $G = \{u < 1\}$.

case 1: $\exists x \in F$ such that $|G \cap B_\rho(x)| \le \epsilon |B_\rho(x)|$. By lemma 3.5, $2u \ge 1$ on $B_{\rho/4}(x)$. By lemma 3.4,

$$\inf_{B_R} u \ge \frac{c}{2} \left(\frac{\rho}{4R} \right)^s.$$

On the other hand, $|B_R \setminus B_{R-\rho}| \ge |E \setminus F| = |E|/2 \ge \theta |B_R|/2$. This implies $\rho \ge R \cdot [1 - (1 - \theta/2)^{1/n}]$ and hence $E_l = B_R$.

case 2: $\forall x \in F \subset G^c$, $|G \cap B_\rho(x)| > \epsilon |B_\rho(x)|$. By Lebesgue theorem, $\frac{|G\cap B_r(x)|}{|B_r(x)|} \to 0$ for almost all $x \in F$. Let $F' = \{x \in F : \frac{|G\cap B_r(x)|}{|B_r(x)|} \to 0\}$. For $x \in F'$, there exists $r(x) > 0$ such that $|G \cap B_{r(x)}(x)| = \epsilon \cdot |B_{r(x)}(x)|$ where $r(x) \in (0, \rho)$. Let $K \subset F'$ be a compact set such that $|K| \geq |F'|/2$. By compactness, we can find finitely many $x_i \in K$ such that

$$K \subset \cup_{i=1}^N B_{r(x_i)}(x_i).$$

We can apply ball covering arguement. Chooses the ball with the largest radius and removes all balls intersecting it. Then select the second largest ball and then throw all balls intersecting it again. Iteriate the process, we obtain a sequence of disjoint balls $B_{r(x_j)}(x_j)$ in which union of $B_{3r(x_j)}(x_j)$ cover K. Noted that $B_{4r(x_j)}(x_j) \subset B_{4R}$ as $|x_j| + 4r(x_j) \leq R - \rho + 4\rho \leq 4R$. Apply lemma 3.6, we conclude that

$$\inf_{B_{r(x_j)}(x_j)} u \geq \gamma.$$

That is to say $B_{r(x_i)}(x_i) \subset E_1$. Therefore,

$$\begin{aligned} |E_1| - |E_0| = |E_1 \setminus E_0| &\geq \sum_j |(E_1 \setminus E_0) \cap B_{r(x_j)}(x_j)| = \sum_j |G \cap B_{r(x_j)}(x_j)| \\ &= \epsilon \cdot \sum_j |B_{r(x_j)}(x_j)| = \frac{\epsilon}{3^n} \cdot \sum_j |B_{3r(x_j)}(x_j)| \\ &\geq \frac{\epsilon}{3^n}|K| \geq \frac{\epsilon}{2 \cdot 3^n}|F'| = \frac{\epsilon}{2 \cdot 3^n}|F| = \frac{\epsilon}{4 \cdot 3^n}|E|. \end{aligned}$$

So $|E_1| \geq (1+\beta)|E_0|$.

This finishes the proof for weak Harnack inequality.

□

4 Full Harnack inequality

In this section, we consider both cases $L = a_{ij}\partial_i\partial_j$ or $L = \partial_i(a_{ij}\partial_j)$ with uniform ellipticity $\lambda > 0$. First we recall the following weak Harnack inequality which holds on both situation as illustrated in the past two sections.

Theorem 4.1. *(weak Harnack) Suppose $Lu = 0$ in B_{4R} and $u \geq 0$, then if we have*

$$|\{u \geq 1\} \cap B_R| \geq \theta \cdot |B_R|$$

for some $\theta > 0$, then $\inf_{B_R} u \geq \delta = \delta(\theta, n, \lambda)$.

Theorem 4.2. *(Harnack inequality) If $Lu = 0$ on B_{2R} with $u \geq 0$, then*

$$\sup_{B_R} u \leq C \inf_{B_R} u$$

for some constant $C = C(n, \lambda) > 0$.

Before we proceed to the proof, we need the following lemmas.

Lemma 4.3. *Suppose $Lu = 0$ with $u \geq 0$ on $B_R(x)$. Let $y \in B_{R/9}(x)$ such that $B_r(y) \subset B_R(x)$ and $r \leq \frac{2}{9}R$. If*

$$|\{u \geq 1\} \cap B_r(y)| \geq \theta \cdot |B_r|$$

for some $\theta > 0$, then $u(x) \geq \left(\frac{r}{R}\right)^s \delta$, for some $s = s(n, \lambda)$ and $\delta = \delta(\theta, n, \lambda)$.

Proof. Observe that $B_{4r}(y) \subset B_R(x)$ as $|x - y| + 4r < \frac{R}{9} + \frac{8R}{9} = R$. Apply weak Harnack inequality on $B_{4r}(y)$, we conclude that

$$\inf_{B_r(y)} u \geq \delta_1 = \delta_1(n, \theta, \lambda) > 0.$$

So $B_r(y) \subset \{u \geq \delta_1\}$ which imply

$$|\{u \geq \delta_1\} \cap B_{2r}(y)| \geq |B_r| = \frac{1}{2^n} \cdot |B_{2r}|.$$

So if $B_{8r}(y) \subset B_R(x)$, we may apply weak harnack again to u/δ_1 on $B_{8r}(y)$ to conclude that

$$\inf_{B_{2r}(y)} u \geq \delta_1 \cdot \delta(n, \lambda) = \delta_1 \epsilon.$$

Noted that ϵ is independent of r. So we may repeat the same argument inductively to deduce that whenever $B_{2^{k+2}r}(y) \subset B_R(x)$,

$$\inf_{B(y, 2^k r)} u \geq \epsilon^k \delta_1.$$

Let N be the maximal integer so that $B_{2^{N+2}r}(y) \subset B_R(x)$. Therefore N satisfies

$$|x - y| + 2^{N+2} r \leq R < |x - y| + 2^{N+3} r.$$

Using $R < |x-y| + 2^{N+3}r$, we know that $x \in B(y, 2^{N+3}r)$. Thus $u(x) \geq \epsilon^N \delta_1$. On the other hand, using the first part of inequality, $N \leq \log_2\left(\frac{R}{r}\right)$. Combining all of them,

$$u(x) \geq \epsilon^N \cdot \delta_1 \geq \epsilon^{\log_2\left(\frac{R}{r}\right)} \delta_1 = \left(\frac{r}{R}\right)^s \delta_1$$

where $s = -\log_2 \epsilon > 0$. □

Lemma 4.4. *Suppose that $Lu = 0$ on $B_{4R}(x)$. If*

$$|\{u \leq 0\} \cap B_R(x)| \geq \theta \cdot |B_R|$$

for some $\theta > 0$, then $\sup_{B(x,4R)} u \geq (1 + \delta) u(x)$.

Proof. If $u(x) \leq 0$, then the inequality is trivially true. So we may assume $u(x) > 0$. Assuming $\sup_{B(x,4R)} u = 1$. Let $v = 1 - u$, $Lv = 0$ and $v \geq 0$. The assumption implies

$$|\{v \geq 1\} \cap B_R(x)| \geq \theta \cdot |B_R|.$$

Weak Harnack inequality implies that

$$\inf_{B(x,R)} v \geq \delta(\theta, n, \lambda).$$

That is equivalently to say $\sup_{B(x,4R)} u \geq (1+\delta)u(x)$. $\square$

Lemma 4.5. *Let $Lu = 0$ on $B(x,R)$, $a \in \mathbb{R}$. Then there exists $\epsilon = \epsilon(n,\lambda) > 0$ such that if*

$$\frac{|\{u > a\} \cap B_R(x)|}{|B_R|} \leq \epsilon,$$

we have $\sup_{B(x,R)} u \geq a + 4(u(x) - a)$.

Remark: If L is of non-divergence form, then it follows directly from Lemma (3.5).

Proof. By subtracting a constant, we may assume $a = 0$. Choose $y \in \{u > 0\}$ and $B_r(y) \subset B_R(x)$ so that

$$\frac{|B_r|}{|B_R|} = 2\epsilon$$

where $r = (2\epsilon)^{1/n}R$. The constant $\epsilon > 0$ is to be determined. By assumption,

$$\frac{|\{u > 0\} \cap B_r(y)|}{|B_r|} \leq \frac{|\{u > 0\} \cap B_R(x)|}{|B_r|} \leq \frac{1}{2}.$$

which implies

$$\frac{|\{u \leq 0\} \cap B_r(y)|}{|B_r|} \geq \frac{1}{2}.$$

By Lemma (4.4), $\sup_{B(y,4r)} u \geq (1+2\delta) \cdot u(y)$ provided $B(y,4r) \subset B(x,R)$. So we have the following conclusion.

Claim: If $B(y,4r) \subset B(x,R)$, then there exists $y' \in B(y,4r)$ such that $u(y') \geq (1+\delta) \cdot u(y)$.

Construct a sequence $\{x_k\}_{k \geq 0}$ as follows. Assume ϵ is sufficiently small so that $B(x,4r) \subset B(x,R)$. Pick $y = x = x_0$ in the place of above claim, we obtain a $x_1 \in B(x_0,4r)$ so that $u(x_1) \geq (1+\delta) \cdot u(x_0)$. If $B(x_1,4r) \subset B(x,R)$, then apply the above claim again to find a $x_2 \in B(x_1,4r)$ in which $u(x_2) \geq (1+\delta) \cdot u(x_1)$. Repeat the same step inductively, we constructed a sequence x_k so that $u(x_{k+1}) \geq (1+\delta) \cdot u(x_k)$ and $|x_{k+1} - x_k| < 4r$. Therefore,

$$u(x_k) \geq (1+\delta)^k \cdot u(x) \quad \text{and} \quad |x_k - x| < 4rk.$$

So x_k exists if $4rk < R$. Let N be the maximum integer so that $4rN < R$. That is $4Nr < R$ but $4(N+1)r \geq R$. Combine all these,

$$\sup_{B(x,R)} u \geq \; u(x_k) \geq (1+\delta)^k u(x) \geq (1+\delta)^{4^{-1}(2\epsilon)^{-1/n}-1} \cdot u(x).$$

The conclusion holds if we choose ϵ is very small depending only on n and $\delta = \delta(n,\lambda)$. $\square$

Instead of proving the full Harnack inequality, we prove a equivalent form first.

Theorem 4.6. *If $Lu = 0$ with $u \geq 0$ on $B(x, 100R)$, then there exists $C = C(n, \lambda) > 0$.*

$$\sup_{B(x,R)} u \leq Cu(x).$$

Proof. Assume $\sup_{B(x,R)} u = 2$. Our objective is to show that $u(x)$ is bounded below by positive constant.

First we construct a sequence $x_k, k \geq 0$ in which $u(x_k) = 2^k$. Since $\sup_{B(x,R)} u = 2$, there exists $x_1 \in \overline{B(x,R)}$ such that $u(x_1) = 2$. Suppose we have constructed $x_k \in B(x, 2R)$, consider

$$r_k = \sup\{r \in (0, R] \ : \ \sup_{B(x_k,r)} u \leq 2^{k+1}\}.$$

r_k must exists as $B(x_k, r) \subset B(x, 3R)$. If $r_k = R$, then we terminate the process. Else, we know that

$$\sup_{B(x_k,r_k)} u = 2^{k+1}.$$

So there exists $x_{k+1} \in \overline{B(x_k, r_k)}$ such that $u(x_{k+1}) = 2^{k+1}$. If $x_{k+1} \notin B(x, 2R)$, we terminate the process as well and ignore the final term x_{k+1}. Therefore, we constructed a sequence $\{x_k\}$ in which $x_k \in B(x, 2R)$, $u(x_k) = 2^k$ and $|x_{k+1} - x_k| \leq r_k$ for all k. Because of the second condition, it can be seen that the sequence can at most finitely many as u is bounded on the compact set $\overline{B(x, 2R)}$. Now we can estimate $u(x)$. By construction,

$$\sup_{B(x_k,r_k)} u \leq 2^{k+1} < 2^{k-1} + 4(2^k - 2^{k-1}) = a + 4[u(x_k) - a].$$

By Lemma(4.5), there exists $\epsilon = \epsilon(n, \lambda) > 0$ such that

$$\frac{|\{u > 2^{k-1}\} \cap B(x_k, r_k)|}{|B(x_k, r_k)|} > \epsilon.$$

Applying Lemma (4.3) to this situation, replacing R by $100R$, putting $y = x_k$ and $r = r_k$. Then we conclude that

$$u(x) \geq \left(\frac{r_k}{100R}\right)^s \delta' 2^{k-1} = \left(\frac{r_k}{R}\right)^s \delta \cdot 2^k \tag{1}$$

for some s, δ depending only on λ, n.

On the other hand, we also have $r_1 + ... + r_N \geq R$. Otherwise $r_N < R$. Then there exists $x_{N+1} \in \overline{B(x_k, r_k)} \setminus B(x, 2R)$. But

$$R > r_1 + ... + r_N \geq |x_{N+1} - x_1| \geq |x_{N+1} - x| - |x - x_1| \geq 2R - R = R.$$

Contradiction occur.

Since $r_1 + ... + r_N \geq R = \sum_{k=1}^{\infty} \frac{R}{k(k+1)}$, there exists $m \in \{1, 2, ..., N\}$ such that $r_m \geq \frac{R}{m(m+1)}$.

Therefore, by putting $k = m$ in the relation (1).

$$u(x) \geq \left(\frac{r_m}{R}\right)^s \delta \cdot 2^m \geq \frac{\delta \cdot 2^m}{m^s(m+1)^s} > C = C(n,\lambda) > 0.$$

$\square$

Corollary 4.7. *Theorem 4.6 implies theorem 4.1, that is the full Harnack inequality.*

Proof. Assume $Lu = 0$ on $B(x, 2R)$ with $u \geq 0$. Let $r = \frac{R}{200}$, then for all $y \in B(x, R)$, $B(y, 100r) \subset B(x, 2R)$. By Theorem (4.6), there exists $C = C(n, \lambda) > 0$ such that

$$\sup_{B(y,r)} u \leq Cu(y)$$

for each $y \in B(x, R)$. Hence for all $p, q \in B(x, R)$ such that $|p - q| < r$, we have

$$u(q) \leq \sup_{B(p,r)} u \leq Cu(p). \tag{2}$$

Now whenever $p, q \in B(x, R)$, we can find a straight line $\gamma : [0, L] \to B(x, R)$ joining from p to q where $L = |p - q|$. So $\gamma(t) = \frac{1}{L}\Big[(L-t)p + tq\Big]$ for $t \in [0, L]$. Divide $[0, L]$ into $0 < r < 2r < ... < Nr < L$ in which $(N + 1)r \geq L$ and denote $p_i = \gamma(ir)$. Using (2), we have $u(p_i) \leq Cu(p_{i+1})$ for $i = 0, 1, ..., N - 1$. Hence

$$u(p) \leq C^{N+1} \cdot u(q) \leq C^{\frac{L}{r}+1} \cdot u(q) \leq C^{101} \cdot u(q).$$

As the inequality holds for arbitrary p, q in $B(x, R)$. This finish the proof.

Remark: We can also use covering lemma's arguement to control the number of balls covering geodesic. $\square$